

Self-force of a rigid ideal fluid, and a charged sphere in hyperbolic motion

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We present two results in the treatment of self-force of accelerating bodies. If the total force on an extended rigid object is calculated from the change of momentum summed over planes of simultaneity of successive rest frames, then we show that an ideal fluid, moving rigidly, exerts no net force on its boundary. Under this same definition of total force, we find the electromagnetic self-force for a spherical charged shell of proper radius R accelerating with constant proper acceleration g is $(2e^2 g/R)[1/12 - \sum_{n=0}^{\infty} (gR)^{2n}((2n-3)(2n-1)(2n+1)^2)^{-1}]$.

Recent work on self-force and radiation reaction in classical electromagnetism has clarified various issues which had been unclear for a considerable period (roughly one hundred years). Several of the difficulties that persisted for a long time were concerned with taking the point-limit for an entity possessing finite charge and observed mass. These problems go away when one insists that there is no such limit, because it is not legitimate to assume the non-electromagnetic contribution to the mass is negative [1–5]. One then has to grapple with the fact that an exact treatment of motion of charged entities cannot deal with point-particles, if the charge is not itself infinitesimal, but must treat extended objects or distributions of charge.

As soon as one has to deal with an extended distribution of charge, the question of its field and motion is no longer a question of finding a single worldline and applying standard formulae to find the field; instead one has to find the shape of a world-tube, and the self-interaction cannot in general be calculated exactly in closed form. Owing to the complexity of the problem, there are few exact general statements than can be made, either about the forces on, or the electromagnetic fields produced by, accelerating charged bodies. In this situation it is useful to identify some statements that have as wide an application as possible, and to obtain some example non-trivial exact solutions.

A modest but useful step of the first kind was taken by Ori and Rosenthal [6], who pointed out that the summed electromagnetic interaction of a pair of point charges of fixed proper separation was independent of the orientation of the line between the charges, as long as it is summed a certain way that we describe in section I. A step of the second kind was taken in [7], where electromagnetic self-force was calculated exactly for a given case. In the present paper we obtain a further general statement about rigid-body motion, namely: the total force exerted by an ideal fluid on its boundary, when the fluid moves rigidly, is zero, for a certain natural definition of this force, namely that adopted in [6, 8, 9]. We then extend the result of [7] by obtaining the electromagnetic self-force of a charged spherical shell in rigid hyperbolic motion, taking the sum over the body in the same sense as recommended in [6, 8, 9]. We also invoke the results of [10] to deduce that the result is applicable to a charged spherical shell at rest in a uniformly accel-

erating reference frame (the Rindler frame) and observed by an observer at rest in that frame.

I. DEFINING THE SELF-FORCE

In relativistic calculations concerning extended bodies, the first issue that arises is, how to define properties such as energy and momentum. This is non-trivial because when the entity is not isolated, the total four-momentum of its parts can depend on the choice of spacelike hypersurface over which they are summed. This dependence does not arise for an isolated body [11], but a charged body is never isolated because it is in permanent interaction with its own electromagnetic field.

This issue was discussed in [6, 8, 9] and somewhat more generally in [10]. Consider a composite object that can be decomposed into a set of discrete entities i . The total 4-momentum of the composite object may be defined as the sum of the 4-momenta of its parts, where the sum is taken over some spacelike hypersurface which we denote by χ :

$$p_{\text{tot}}^\mu(\tau_c, \chi) = \sum_i p_i^\mu(\tau_{i,\chi}). \quad (1)$$

$\tau_{i,\chi}$ is the proper time on the i 'th worldline when that worldline intersects χ , and τ_c is the proper time on some reference worldline (e.g. the worldline of the centroid). Typically, one picks a spacelike hyperplane (so that the events $\{i\}_\chi$ are simultaneous in some frame). In general, p_{tot}^μ depends on χ so this 'total 4-momentum' is not a property of the object alone, and it will not behave like a 4-vector unless we specify χ in a suitably covariant way. A suitable way, is, for example, to pick χ such that it is the same surface, irrespective of what frame may be used to calculate the four components of each p_i .

For an object whose motion is rigid—that is, its motion is such that at any given event on the world-tube there is a reference frame in which all parts of the object are at rest, and at the same proper distances—a natural choice of χ is the hyperplane of simultaneity for the instantaneous rest frame (IRF) at the given τ_c . By making this choice we obtain a well-defined 4-vector p_{tot}^μ , but it is not the only possible choice. *Any* recipe that picks out a unique hyperplane χ for each τ_c will result in a p_{tot}^μ that transforms in the right way, because the terms

in the sum on the right hand side of (1) all do, and the recipe fixes the set of events in a frame-independent way. As a result, when calculating the components of p_{tot}^μ relative to any given inertial frame at some given τ_c , one may be summing over events that are not simultaneous in that frame, but one must accept this in order to have a well-defined 4-vector.

If there are several bodies in different states of motion in a given problem, there will not be a unique IRF for all of them, so the policy of adopting the IRF in order to define χ is not necessarily the only policy that makes sense.

Once we have decided how to choose χ for each τ_c , it becomes possible to define the total rate of change of 4-momentum:

$$\frac{dp_{\text{tot}}^\mu}{d\tau_c} = \lim_{\delta\tau_c \rightarrow 0} \frac{p_{\text{tot}}^\mu(\tau_c + \delta\tau_c, \chi + \delta\chi) - p_{\text{tot}}^\mu(\tau_c, \chi)}{\delta\tau_c} \quad (2)$$

where we have assumed a one-to-one correspondence between χ and τ_c , such that $\delta\chi \rightarrow 0$ as $\delta\tau_c \rightarrow 0$. Hence [6, 10]

$$\frac{dp_{\text{tot}}^\mu}{d\tau_c} = \sum_i \frac{dp_i^\mu}{d\tau_i} \frac{d\tau_i}{d\tau_c} \quad (3)$$

where each $d\tau_i$ is the proper time elapsed on the i 'th worldline between the intersections of that worldline with χ and $\chi + d\chi$, and the quantities $dp_i^\mu/d\tau_c$ and $d\tau_i/d\tau_c$ are evaluated on the hyperplane χ .

In [7] the case of a rigid spherical shell was considered, with χ taken as the hyperplane of simultaneity of the IRF at some instant, and $\chi + d\chi$ taken as a parallel hyperplane such that $d\tau_i = d\tau_c$. This is a legitimate choice, as we made clear above, but it is not the only one that makes sense, and another case is interesting, namely, to take for $\chi + d\chi$ the new hyperplane of simultaneity, i.e. the one associated with the next state of motion, which is not the same as the initial one when the body is accelerating. This is the choice made by Nodvik [8] and also recommended in [6, 9]. Therefore we will explore its impact on the case of the sphere in rigid hyperbolic motion in section III. First we apply it to obtain a useful observation concerning internal stress.

II. SELF-FORCE OF A RIGID IDEAL FLUID

We consider the net force on a rigid body owing to its internal stress.

Rigid motion as defined above (at each moment there is an IRF for the whole body, and the proper shape and size does not change) implies motion in a straight line, but the acceleration need not be constant. Such motion is treated in section II of [8] and section 9.2.1 of [11]. The condition for rigidity is that worldlines at points A and B on the body should satisfy Eqs (9.23) and (9.24) of [11], which we reproduce here for convenience:

$$x_B - x_A = \gamma L_0, \quad t_B - t_A = \gamma v L_0 \quad (4)$$

where x_A, t_A, x_B, t_B are coordinates in an inertial frame, L_0 is the proper separation of x_B and x_A , $v = dx_A/dt_A$ and $\gamma = (1 - v^2)^{-1/2}$ (taking $c = 1$). From these equations we find

$$\frac{d^2 x_B}{dt_B^2} = \frac{\dot{v}}{1 + \dot{v} L_0} \quad (5)$$

where \dot{v} is the proper acceleration on worldline A. Now, by definition, for rigid motion, there is always an IRF. If we evaluate (5) in this frame, then the quantity on the left hand side is the proper acceleration on worldline B, and $L_0 = x_B - x_A$. Then we have

$$\frac{d^2 x_B}{dt_B^2} = \frac{x_A - x_0}{x_B - x_0} \dot{v} \quad (6)$$

where $x_0 = x_A - 1/\dot{v}$. Hence we have that for rigid motion in the x direction, there exists a plane $x = x_0$ in the IRF, such that the acceleration of each part of the body is inversely proportional to its distance from the plane.

Now suppose the interior of the body in question behaves as an ideal fluid, i.e. it can support pressure or tension but not sheer stress. For such a fluid, in the IRF the Navier-Stokes equation takes the form

$$(\rho_0 + p) \frac{D\mathbf{u}}{Dt} = -\nabla p \quad (7)$$

where p is the pressure and ρ_0 the mass density. Choose the origin such that $x_0 = 0$, then, using (6), the acceleration $D\mathbf{u}/Dt \propto \hat{\mathbf{x}}/x$ and therefore the pressure satisfies the differential equation

$$\frac{dp}{dx} = -\frac{\rho_0 + p}{x} \quad (8)$$

whose solution is

$$p = \frac{\text{const}}{x} - \rho_0. \quad (9)$$

The total force exerted by the ideal fluid on its boundary is, using (3),

$$\mathbf{f}_p = \oint \frac{d\tau_x}{d\tau_c} p d\mathbf{S}. \quad (10)$$

We now make our choice of the hyperplanes χ and $\chi + d\chi$ in order to evaluate $d\tau_x/d\tau_c$. We adopt the hyperplanes of simultaneity of the two successive IRFs, which leads to

$$\frac{d\tau_x}{d\tau_c} = \dot{v} x \quad (11)$$

where \dot{v} is the proper acceleration at $x = x_c$. The proof of this equation is given in the next section. Assuming it for the present argument, we have

$$\mathbf{f}_p = \oint \dot{v} x p d\mathbf{S}. \quad (12)$$

Let \mathbf{e}_i be a unit vector in the i 'th direction. Then the component of the force in the i 'th direction is

$$\mathbf{f}_p \cdot \mathbf{e}_i = \oint \dot{v} x p \mathbf{e}_i \cdot d\mathbf{S} = \int \nabla \cdot (\dot{v} x p \mathbf{e}_i) dV. \quad (13)$$

Now,

$$\nabla \cdot (\dot{v} x p \mathbf{e}_i) = \dot{v} \left(p + x \frac{dp}{dx} \right) \delta_{1i} = -\dot{v} \rho_0 \delta_{1i} \quad (14)$$

using (8). Hence

$$\mathbf{f}_p = -m_0 \mathbf{a} \quad (15)$$

where \mathbf{a} is the proper acceleration on the reference worldline, and $m_0 = \int \rho_0 dV$ is the ‘completely bare’ mass, i.e. the mass of the fluid without taking either the stress or any electromagnetic contribution into account. Hence we have that, *for rigid motion of an ideal fluid, the pressure or tension in the fluid gives no net force on its boundary, when it is defined as the rate of change of total momentum evaluated in the sequence of IRFs.* Alternatively, the same result may be stated as that the contribution to the inertial mass that we might have expected the stress to give, namely $\int p dV$, is exactly balanced by the difference in pressure across the body.

We have not needed to make any assumption about the shape of the body in order to derive equation (15). It applies equally to a dipole as to a sphere, for example. In particular, it shows that the net force owing to internal pressure does not depend on the orientation of the body relative to its acceleration, when it is evaluated using the sequence of IRFs to define the hyperplanes on which the forces are summed (the forces being conveniently summed by appropriate use of Eq. (3)). This gives a further reason, in addition to the one obtained in [6], to recommend this choice when discussing self-force for bodies moving rigidly.

III. ELECTROMAGNETIC SELF-FORCE OF A RIGID SPHERICAL SHELL IN HYPERBOLIC MOTION

The electromagnetic self-force of a uniformly accelerating rigid charged spherical shell was calculated in [7]. That calculation gave the result of summing the electromagnetic force over such a sphere in its instantaneous rest frame. The force was taken to be given by the sum of the local forces on each element of charge, evaluated at events simultaneous in the rest frame. In our present notation, this amounts to taking $\chi + d\chi$ parallel to χ and hence $d\tau_1 = d\tau_c$ in Eq. (3). This is a legitimate choice, but not necessarily the most useful one. As we have seen in the previous section, and also for the reasons given in [6, 8], it is useful to consider also the case where $\chi + d\chi$ is the hyperplane of simultaneity of the next IRF. This is the subject of this section.

An object undergoing rigid hyperbolic motion in the x -direction will be at rest relative to the constantly accelerating reference frame described by the Rindler metric:

$$g_{ab} = \text{diag}(-x^2, 1, 1, 1). \quad (16)$$

The line element is

$$ds^2 = -x^2 dt^2 + dx^2 + dy^2 + dz^2. \quad (17)$$

For this metric, the local ‘acceleration due to gravity’ (that is, the acceleration, relative to the coordinate frame, of an object in free fall) is $-1/x$.

Lines at constant (x, y, z) are worldlines of points on a body at rest in the frame. Hyperplanes at constant t are planes of simultaneity for the successive IRFs. The proper time between events on two such planes at any given (x, y, z) is given by

$$d\tau_x = x dt. \quad (18)$$

Therefore

$$\frac{d\tau_x}{d\tau_c} = \frac{x}{x_c} = gx, \quad (19)$$

where τ_c is proper time along a reference worldline which we take as the one at $(x, y, z) = (x_c, 0, 0)$, and $g = 1/x_c$ is the proper acceleration on this reference worldline. Hence for this case, Eq. (3) reads

$$\frac{dp_{\text{tot}}^\mu}{d\tau_c} = g \sum_i \frac{dp_i^\mu}{d\tau_i} x_i. \quad (20)$$

The same argument can be used to obtain the slightly more general Eq. (11), in which \dot{v} is not necessarily constant. This is done by adopting the Rindler frame matched to the instantaneous value of \dot{v} , for a body whose acceleration can change.

We now apply this to find the electromagnetic contribution to the self-force of a spherical shell of charge undergoing rigid hyperbolic motion. This self-force is given by summing the force $d\mathbf{f}$ on each element of charge dq making up the shell, where $d\mathbf{f}$ is owing to the electromagnetic field sourced by the rest of the shell. The calculation follows the argument of [7] closely, so we shall refer to that paper as PRS, and its equations as (PRS n) where n is the equation number.

In the case under consideration, the charge is concentrated in a thin spherical shell. Therefore, by the argument leading to Eq. (PRS1.4) we have

$$\mathbf{f}_{\text{self}} = \oint g x \sigma \frac{\mathbf{E}_- + \mathbf{E}_+}{2} dS \quad (21)$$

where $\mathbf{E}_- = \lim_{\epsilon \rightarrow 0} \mathbf{E}(R - |\epsilon|)$ is the field on the interior surface of the shell, and $\mathbf{E}_+ = \lim_{\epsilon \rightarrow 0} \mathbf{E}(R + |\epsilon|)$ is the field on the exterior surface of the shell. R is the radius of the shell. By standard reasoning from Maxwell’s equations in an inertial frame we have $\mathbf{E}_+ = \mathbf{E}_- + (\sigma/\epsilon_0)\hat{\mathbf{r}}$, therefore

$$\mathbf{f}_{\text{self}} = \sigma \oint g x \mathbf{E}_- dS + \mathbf{f}_\sigma \quad (22)$$

where

$$\mathbf{f}_\sigma = \oint \frac{\sigma^2}{2\epsilon_0 R} \mathbf{r} g x dS = \frac{e^2 g}{6R} \hat{\mathbf{x}} \quad (23)$$

where $e^2 = q^2/4\pi\epsilon_0$ (in SI units) if q is the total charge of the shell, and \mathbf{r} is the vector $(x - x_c, y, z)$.

To calculate the integral in (22), note that only the x -component is non-zero, and use

$$E_{-,x}(x, \rho) = \sum_{n=0}^{\infty} a_n E_{x,n}(x, \rho) \quad (24)$$

where $E_{x,n}(x, \rho)$ is given by equation (PRS4.17) after suitably scaling the units, and the calculation of the coefficients a_n is described in PRS. We find

$$\begin{aligned} f_{\text{self},n} &= 2\pi R \sigma \int_{(1/g)-R}^{(1/g)+R} g x E_{x,n}(x, \rho(x)) dx \\ &= \frac{2ge^2}{R} \left\{ (-1)^{n+1} + \sum_{m=1}^n \frac{(-1)^{m+n} \binom{n}{m} (gR)^{2m}}{(2m-1)(2m+1)} \right\}. \end{aligned} \quad (25)$$

The self-force is then given by

$$f_{\text{self}} = f_\sigma + \sum_{n=0}^{\infty} a_n f_{\text{self},n} \quad (26)$$

The end result is that the electromagnetic self-force of the uniformly charged spherical shell undergoing rigid hyperbolic motion with proper acceleration g is

$$f_{\text{self}} = \frac{ge^2}{R} \left(\frac{1}{6} - 2 \sum_{n=0}^{\infty} \frac{(gR)^{2n}}{(2n-3)(2n-1)(2n+1)^2} \right) \quad (27)$$

where I have calculated this result for terms up to order R^{101} , and I conjecture its validity at all orders. The result differs from (PRS5.11) because of the different choice of hyperplane $\chi + d\chi$, as we have explained above. At $g = 1$ and $e^2 = 1$ the first few terms in this series are

$$f_{\text{self}} \simeq \frac{-1}{2R} + \frac{2}{9}R - \frac{2}{75}R^3 + \frac{2}{735}R^5 + \frac{2}{2835}R^7 + \dots \quad (28)$$

Eq. (28) agrees with Nodvik [8], who considered a shell undergoing arbitrary motion and calculated the lowest order terms in the series expansion. For the case of hyperbolic motion Nodvik's result (Eqs (7.21)-(7.24) of [8]) is

$$f_{\text{self}}^{\text{Nodvik}} = -\frac{1}{2R} + \frac{2}{9}R + O(R^3), \quad (29)$$

Nodvik formulated the problem in such a way that he calculated the self-force using a prescription that corresponds to our Eq. (20), so we expect his result to agree

with ours, as it does. The reason for the disagreement with (PRS5.11) was wrongly described in ([7]), where it was attributed to the effect of internal stress. We can now understand this better through the insight described in the previous section, leading to Eq. (15). When the momenta are summed in successive IRFs, a calculation which neglects to include the effects of internal stress will nevertheless arrive at a 4-vector force with the correct Lorentz transformation properties, because the pressure does not appear in Eq. (15). In short, the '4/3 problem' does not arise. But this does not mean that it is correct to neglect the stress. In fact, only when it is correctly included does one arrive at Eq. (15).

A. Electromagnetic self-force in the Rindler frame

So far we have only discussed forces observed by inertial observers. We made use of the Rindler metric in order to obtain Eq. (19), but (27) is the self-force observed by an inertial observer, relative to whom the sphere accelerates. We now briefly comment on the electromagnetic self-force observed by an observer at rest relative to the sphere; that is, an accelerated observer. Such an observer's measurements of space and time are indicated by the coordinates (t, x, y, z) appearing in the line element (17), and therefore his most natural definition of total force on an extended object at rest relative to him is the one given by (20).

If he so chooses, inertial forces may be considered by the accelerated observer as an example of a gravitational field. Such an observer finds that the field-lines of the sphere 'droop' in the gravitational field, and consequently the sphere exerts once again a self-force, owing to the fact that the electric forces between all the pairs of charges composing the sphere do not sum to zero. It was shown in [10] that the electromagnetic force observed by such an accelerated observer is the same as the one observed by the inertial observer who is at rest in the IRF of the sphere. Consequently it is given by (27).

To conclude, the main results of this paper are Eqs (15) and (27). (15) is a general statement about rigid motion of a body whose internal forces correspond to those of an ideal fluid. The shape and the variation of the acceleration with time are arbitrary. Eq. (27) is a correct to high order, and conjectured exact, statement of the electromagnetic self-force of a rigid spherical shell undergoing hyperbolic motion. It is a partner to Eq. (5.11) of [7]. The two equations differ because there is no unique definition of 'the' self-force, owing to the dependence on the choice of hypersurfaces in spacetime over which the momenta are summed. However, the choice made in the present paper is arguably the more neat or natural. The previous study enabled the fields to be found exactly, no matter how the total force is defined, and gave the self-force under another reasonably natural definition.

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